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Crack deflection by an interface – asymptotics of the residual thermal stresses

D. Leguillon [a,*], C. Lacroix [a], E. Martin [b]

[a] *Laboratoire de Modélisation en Mécanique, CNRS UMR 7607, Université Pierre et Marie Curie, Paris 6, 8 rue du Capitaine Scott, 75015 Paris, France*

[b] *Laboratoire de Génie Mécanique, UPRES 496, IUT A, Université de Bordeaux I, 33405 Talence, France*

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Abstract

The He and Hutchinson (1989) criterion allows to predict a matrix crack deflection by an interface under mechanical loadings. It has been improved by the same authors and Evans (1994) to take into account the residual thermal stresses. This criterion is revisited here from an asymptotic point of view. A first-order (local) and second-order (non-local) forms of this criterion are proposed for mechanical loadings. Next, the same approach is carried out for a pure thermal loading, i.e. for a cooling process for instance. It is shown that residual thermal stresses do not intervene at the leading order but play a role at the second one. When combined loadings are considered, the first-order criterion is still unchanged but the second-order one becomes quite illusory since it makes an explicit reference to the critical mechanical loading triggering the deflection. Nevertheless, some trends on the deflection promotion or inhibition are derived for a notched bimaterial under four-point bend loading. In the light of these results, criticisms to the He et al. (1994) expressions are brought. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Composite materials; Residual thermal stresses; Interfaces; Crack deflection

1. Introduction – the He and Hutchinson deflection criterion

Brittleness of new materials like ceramics is often an undesired counterpart to their high mechanical and thermal qualities. To avoid these negative effects, fibres or other inclusions are inserted, not as reinforcements (their stiffness is often lower than that of the bulk material), but in order to promote toughening processes (Evans, 1997). Cracks growing within the matrix impinge on the interface and are expected to kink along the interfaces with these inclusions and either to blunt the primary crack tip (T-crack, Benveniste et al., 1989; Dollar and Steif, 1992) or to develop dissipative processes by friction.

* Corresponding author. Address: Laboratoire de Modélisation en Mécanique, CNRS UMR 7607, Université Pierre et Marie Curie, Tour 66 - Case 162, 4, Place Jussieu, 75252 Paris, Cedex 05, France. Tel.: +33-1-4427-5322; fax: +33-1-4427-5259.

E-mail addresses: lmm@cicrp.jussieu.fr, dol@ccr.jussieu.fr (D. Leguillon).

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An efficient criterion able to predict such crack deflection is of course essential to taylor these composite materials. An approach has been proposed by He and Hutchinson (1989) (see also Gupta et al. (1992), Martinez and Gupta (1994) and Kumar and Singh (1998)), it is based on the classical analysis of energy release rates at the tip of virtual crack extensions either deflecting along the interface or penetrating into the fibre, once the primary crack impinges on the interface. Mechanical loadings are considered in such mechanisms but because of the thermal treatments involved in the manufacturing of these composite materials, the influence of residual thermal stresses which result of a cooling process must not be omitted. In view of these remarks, He et al. (1994) improved the previous criterion by additional terms taking into account residual thermal stresses.

In the present analysis, asymptotics show that residual stresses intervene in two ways. First, the intensity factors split into a mechanical part (analogous to the SIF's) and a thermal one (analogous to the TSIF's, Meyer and Schmauder, 1992). Second, the asymptotics of the displacement fields include an additional non-singular term (Munz et al., 1993) which is of second order, it means that it is negligible provided the virtual increments keep very small. If not, it is necessary to retain this second-order influence as well as the mechanical one which lacks in the above mentioned analysis by He et al. (1994)). Second-order, either thermal or mechanical, terms modify substantially the criteria, simplifications are no longer allowed and the role of increment lengths is enhanced, although it was not totally absent in the initial forms as will be seen below. Moreover, the criteria evolve from a local form at the leading order to a non-local one if the second-order terms are employed.

1.1. The He and Hutchinson deflection criterion

We consider a plane strain linear elasticity problem for a bimaterial Ω made of two isotropic layers $\Omega^{(k)}(k=1,2)$ defined by the Young's moduli $E^{(1)}$ and $E^{(2)}$ (Fig. 1). For simplicity, it is assumed that the two materials have the same Poisson ratio $\nu^{(1)}=\nu^{(2)}=\nu$.

Material 1 is pre-cracked, the slit lies from the edge to the interface with the second material in the middle of the specimen. The structure is submitted to a four-point bending test which is symmetrical and allows computational as well as analytical simplifications. The equations of the problem, settled on the half structure still denoted Ω (Fig. 2), read

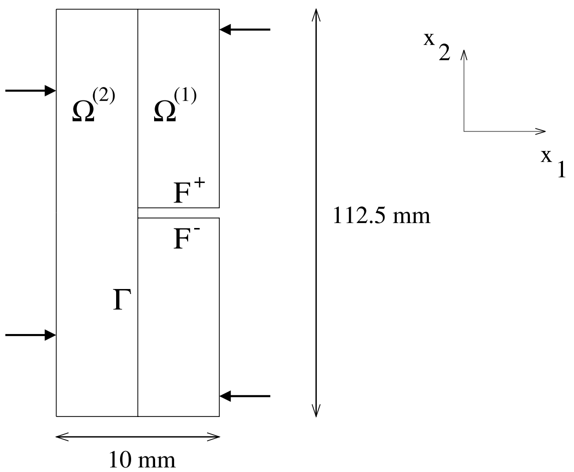

Fig. 1. The specimen under four-point bend loading.

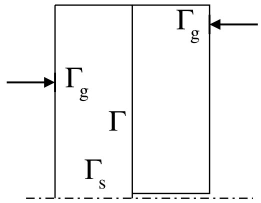

Fig. 2. The half structure obtained by symmetry.

$$\begin{cases} -\nabla\cdot\sigma^{\mathrm{m}} = 0 \quad \text{in } \Omega = \Omega^{(1)}\cup\Omega^{(2)}, \\ \sigma^{\mathrm{m}} = C\cdot\nabla_{\mathrm{s}}\underline{U}^{\mathrm{m}} \quad \text{in } \Omega, \\ \sigma^{\mathrm{m}}\cdot\underline{n} = g\underline{n} \quad \text{on } \Gamma_g, \\ \sigma^{\mathrm{m}}_{12} = 0, \quad U^{\mathrm{m}}_2 = 0 \quad \text{on } \Gamma_{\mathrm{s}}, \\ \sigma^{\mathrm{m}}\cdot\underline{n} = 0 \quad \text{elsewhere on } \partial\Omega, \end{cases} \tag{1}$$

where $\underline{U}^{\mathrm{m}}$ is the displacement field, σ^{m} the stress field (the index m distinguishes mechanical from thermal loadings). They are linked together by the constitutive law (1) (second term), ∇_{s} is the symmetrical part of the gradient, C is the elasticity tensor, it takes different values in $\Omega^{(1)}$ and $\Omega^{(2)}$ depending on $E^{(k)}(k = 1,2)$ and ν. Eq. (1) (third to fifth terms) are the boundary conditions, g is the pressure applied on the specimen at the loading points of the bending test and Eq. (1) (fourth term) are symmetry conditions. The last relation (1) (fifth term) expresses that the remaining part of the boundary $\partial\Omega$ of the specimen, including the faces F^+ and F^- of the primary crack, is stress free. Here and throughout the paper $\underline{n}$ denotes a unit (outer) normal to the line (2D) under consideration.

In order to study the conditions for a crack impinging on the interface to deflect along the interface (index d) or to penetrate into material 2 (index p), He and Hutchinson (1989) consider two crack increments with respective length a_{d} and a_{p} which are assumed to be small (Fig. 3). Then, using integral formulations, they estimate the energy release rate at the new crack tips, either in deflection G_{d} or in penetration G_{p}. The ratio $G_{\mathrm{d}}/G_{\mathrm{p}}$, which arises to be independent of the applied loads, is compared to the ratio of the interface toughness $G_{i\mathrm{c}}$ to the material 2 toughness $G_{2\mathrm{c}}$. Deflection is assumed to occur if

$$\frac{G_{\mathrm{d}}}{G_{\mathrm{p}}} > \frac{G_{i\mathrm{c}}}{G_{2\mathrm{c}}}.$$

The left hand side takes the particular form

$$\frac{G_{\mathrm{d}}}{G_{\mathrm{p}}} = R\left(\frac{a_{\mathrm{d}}}{a_{\mathrm{p}}}\right)^{2\lambda-1},$$

where R is a geometrical parameter and λ the singularity exponent at the tip of the primary crack. Nevertheless more thorough explanations will be given below and the discussion will not take place here. In

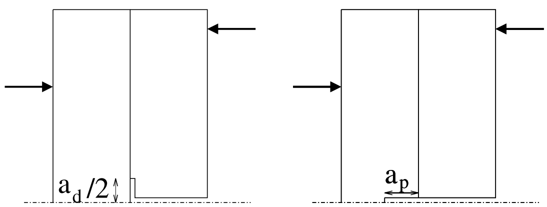

Fig. 3. The crack increment in deflection and penetration.

particular, He and Hutchinson add, for obvious reasons, the questionable assumption $a_d = a_p$ which is debated in Ahn et al. (1998). The choice of the increment lengths is still an open problem. We have recently proposed an argument based on the minimum of total energy principle of Lawn to derive these lengths instead of prescribing them (Leguillon et al., 2000a,c). It is found that they differ slightly, but their influence on the deflection criterion vanishes if there is no elastic contrast between the materials and remains small if the elastic mismatch increases.

1.2. First-order matched asymptotics – the HH criterion revisited

A generic crack increment is denoted F_E with faces F_E^+ and F_E^- (Fig. 4). Its dimensionless length ε is the ratio between its actual length and the primary crack length ℓ

$$\varepsilon = \frac{|F_E|}{\ell}, \quad \text{in particular } \varepsilon_{d/p} = \frac{a_{d/p}}{\ell}.$$

As in the He and Hutchinson analysis, ε is assumed to be small. Prior to any crack increment the solution to Eq. (1) is now denoted $\underline{U}^{m0}$ where the index 0 holds for a zero increment length ($\varepsilon = 0$). To solve the perturbed problem (i.e. including the small extension), one has to add to Eq. (1) new boundary conditions on the crack extension faces

$$\sigma(\underline{U}^{m\varepsilon}) \cdot \underline{n} = 0 \quad \text{on } F_E^+ \text{ and } F_E^-.$$

The unknown solution $\underline{U}^{m\varepsilon}$ can be described by means of two expansions. The outer one, valid out of a region surrounding the crack tip and the increment, provides informations on the far displacement field

$$\underline{U}^{m\varepsilon}(x_1,x_2) = \underline{U}^{m0}(x_1,x_2) + f_1(\varepsilon)\underline{U}^{m1}(x_1,x_2) + \cdots \qquad \text{with } \lim_{\varepsilon\to 0} f_1(\varepsilon) = 0. \tag{2}$$

It is a perturbation to the initial solution $\underline{U}^{m0}$. Terms $\underline{U}^{m0}$ and $\underline{U}^{m1}$ are defined in the unperturbed domain Ω. On the other hand, the near displacement field and the inner expansion are obtained after having stretched the perturbed domain by $1/\varepsilon$. The new variables are denoted $y_j = x_j/\varepsilon$ ($j = 1,2$). The resulting domain Ω^{in} is unbounded and the dimensionless stretched increment length is one whatever its actual value. The inner expansion reads

$$\underline{U}^{m\varepsilon}(\varepsilon y_1, \varepsilon y_2) = F_0(\varepsilon)\underline{V}^0(y_1,y_2) + F_1(\varepsilon)\underline{V}^1(y_1,y_2) + \cdots,$$

with

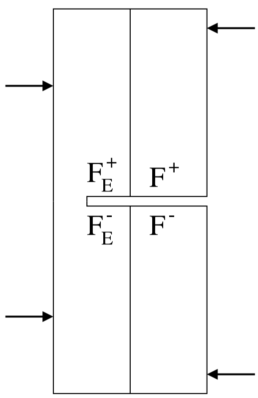

Fig. 4. The two faces of the crack extension.

$$\lim_{\varepsilon \to 0} \frac{F_1(\varepsilon)}{F_0(\varepsilon)} = 0.$$

These two expansions describing the same solution, it must exist an intermediate area in which both expressions coincide. This common area is near the crack tip in the outer domain and far from it in the inner one. The matching conditions are derived from this ascertainment. The behaviour of $\underline{U}^{m0}$ when approaching the crack tip is described at the first-order by the singular field $r^\lambda \underline{u}(\varphi)$ (the first term $\underline{U}^{m0}(0,0)$ is present for consistency but does not play any role)

$$\underline{U}^{m0}(x_1, x_2) = \underline{U}^{m0}(0,0) + k_m r^\lambda \underline{u}(\varphi) + \cdots, \tag{3}$$

where r and φ are the polar coordinates with origin at the primary crack tip (Fig. 5). The coefficient k_m is the intensity factor of the singular mode (analogous to a SIF), it is independent of the geometry of the perturbation but, on the other hand, it depends on the applied loads and the geometry of the structure. A single symmetrical mode is involved here for simplicity (the four-point bending test is symmetrical); however, it exists also an antisymmetrical mode with the same exponent λ which is not activated here (vanishing intensity factor). The situation is quite similar to that of modes I and II at a crack tip in homogeneous materials. The opening condition writes $k_m > 0$ and holds true for the four-point bending test. The exponent λ depends on the relative stiffness of the components, $1/2 < \lambda < 1$ if $E^{(1)} < E^{(2)}$, $0 < \lambda < 1/2$ if $E^{(1)} > E^{(2)}$, the limit $\lambda = 1/2$ corresponds to the classical case of a crack in a homogeneous material $E^{(1)} = E^{(2)}$ (see Section 5). Each term of Eq. (3) can be known at least numerically (Leguillon and Sanchez-Palencia, 1987).

The matching rules lead to

$$\begin{cases} F_0(\varepsilon) = 1, \ F_1(\varepsilon) = k_m \varepsilon^\lambda, \\ \underline{V}^0(y_1, y_2) = \underline{U}^{m0}(0,0), \\ \underline{V}^1(y_1, y_2) \sim \rho^\lambda \underline{u}(\varphi) \qquad \text{as } \rho \to \infty. \end{cases}$$

Here $\rho = r/\varepsilon$ is the stretched radial coordinate. As before, the constant leading term plays no role, moreover function $\underline{V}^1$ has no index m since it is independent of the way loads are applied. The behaviour at infinity of $\underline{V}^1$ is ensured by splitting and superimposition

$$\underline{V}^1(y_1, y_2) = \rho^\lambda \underline{u}(\varphi) + \hat{\underline{V}}^1(y_1, y_2). \tag{4}$$

The complementary term $\hat{\underline{V}}^1$ is the solution of a well-posed problem and decays at infinity. Thus we have

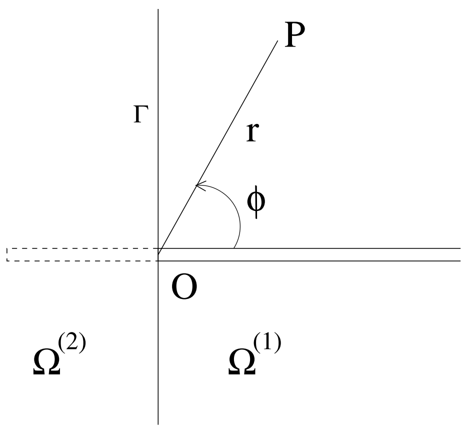

Fig. 5. The polar coordinates with origin at the primary crack tip.

$$\begin{cases} \underline{U}^{\mathrm{m}0}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{\mathrm{m}0}(0,0) + k_{\mathrm{m}}\varepsilon^{\lambda}\rho^{\lambda}\underline{u}(\varphi) + \cdots, \\ \underline{U}^{\mathrm{m}\varepsilon}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{\mathrm{m}0}(0,0) + k_{\mathrm{m}}\varepsilon^{\lambda}[\rho^{\lambda}\underline{u}(\varphi) + \underline{\hat{V}}^1(y_1,y_2)] + \cdots \end{cases} \tag{5}$$

The two terms of the inner expansions are sufficient to give a first-order approximation of the energy release rate. We need only a more precise knowledge of the behaviour of $\underline{\hat{V}}^1$ at infinity, it is defined in Eq. (6) by the so-called dual mode $\rho^{-\lambda}\underline{u}^-(\varphi)$ to $\rho^{\lambda}\underline{u}(\varphi)$ (Leguillon and Sanchez-Palencia, 1987). The expansion of the elastic solution in positive powers of the distance r to the crack tip is a generalization of the Williams series (exponents are not necessary integers and half integers). The successive terms have a finite energy in the vicinity of the tip. On analogy, the behaviour at infinity is described by similar series but with negative powers in order to have a bounded energy at infinity

$$\underline{\hat{V}}^1(y_1,y_2) \sim K\rho^{-\lambda}\underline{u}^-(\varphi) \qquad \text{as } \rho \to \infty. \tag{6}$$

The coefficient K is the intensity factor of the dual mode, it is independent of the geometry of the whole structure and the applied loads, especially it is independent of whether it is a mechanical or a thermal loading. On the other hand it depends on the geometry of the perturbation (the direction of the crack increment in the present case). The dual mode is symmetrical as well, in the general case one has to account also for the dual mode to the antisymmetrical primal one.

Although it is not necessary in this section, it will prove useful in the forthcoming analysis to use once more the matching rules. It brings to determine the next term of the outer expansion (2)

$$f_1(\varepsilon) = k_{\mathrm{m}}K\varepsilon^{2\lambda}, \qquad \underline{U}^{\mathrm{m}1}(x_1,x_2) = r^{-\lambda}\underline{u}^-(\varphi) + \underline{\hat{U}}^{\mathrm{m}1}(x_1,x_2). \tag{7}$$

The complementary term $\underline{\hat{U}}^{\mathrm{m}1}(x_1,x_2)$ is the solution of a well-posed problem in the unperturbed half domain Ω (to compare to Eq. (1))

$$\begin{cases} -\nabla\cdot\sigma^{\mathrm{m}1} = 0 \quad \text{in } \Omega, \\ \sigma^{\mathrm{m}1} = C\cdot\nabla_{\mathrm{s}}\underline{\hat{U}}^{\mathrm{m}1} \quad \text{in } \Omega, \\ \sigma^{\mathrm{m}1}_{12} = 0, \qquad \hat{U}^{\mathrm{m}1}_2 = 0 \quad \text{on } \Gamma_{\mathrm{s}}, \\ \sigma^{\mathrm{m}1}\cdot\underline{n} = -\sigma(r^{-\lambda}\underline{u}^-(\varphi))\cdot\underline{n} \text{ elsewhere on } \partial\Omega. \end{cases} \tag{8}$$

Although the dual mode $r^{-\lambda}\underline{u}(\varphi)$ has not a finite energy in the vicinity of the primary crack tip, the final boundary condition (8) (fourth term) is not questionable since the right hand side vanishes in this vicinity and is smooth out of it.

A necessary condition for the crack to grow by an increment $\varepsilon\ell$ is that the energy consumed during the fracture process $G_{\mathrm{c}}\varepsilon\ell$ (where G_{c} is the toughness of the material) at most equals the energy released between the initial and final states

$$\Delta W > G_{\mathrm{c}}\varepsilon\ell. \tag{9}$$

The calculation of the change in the stored energy reduces to a contour integral ψ

$$\begin{aligned} \Delta W &= W^0 - W^{\varepsilon} \\ &= 1/2\int_{\Omega} C\cdot\nabla_{\mathrm{s}}\underline{U}^{\mathrm{m}0}\cdot\nabla_{\mathrm{s}}\underline{U}^{\mathrm{m}0}\,\mathrm{d}x - \int_{\Gamma_g} g\underline{n}\cdot\underline{U}^{\mathrm{m}0}\,\mathrm{d}l - 1/2\int_{\Omega^{\varepsilon}} C\cdot\nabla_{\mathrm{s}}\underline{U}^{\mathrm{m}\varepsilon}\cdot\nabla_{\mathrm{s}}\underline{U}^{\mathrm{m}\varepsilon}\,\mathrm{d}x + \int_{\Gamma_g} g\underline{n}\cdot\underline{U}^{\mathrm{m}\varepsilon}\,\mathrm{d}l \\ &= 1/2\int_L [\sigma(\underline{U}^{\mathrm{m}\varepsilon})\cdot\underline{n}\cdot\underline{U}^{\mathrm{m}0} - \sigma(\underline{U}^{\mathrm{m}0})\cdot\underline{n}\cdot\underline{U}^{\mathrm{m}\varepsilon}]\,\mathrm{d}l = \psi(\underline{U}^{\mathrm{m}\varepsilon},\underline{U}^{\mathrm{m}0}). \end{aligned} \tag{10}$$

The line L is any contour surrounding the crack tip and the crack increment and starting and finishing on the stress-free faces of the primary crack. Among others, the crack extension faces F_E^+ and F_E^- form an admissible contour which allows to rewrite Eq. (10) as a work done along F_E and gives rise to the COD method (Rybicki and Kanninen, 1977)

$$\Delta W = 1/2\int_{F_E}[\sigma(\underline{U}^{\mathrm{m}\varepsilon})\cdot\underline{n}\cdot\underline{U}^{\mathrm{m}0} - \sigma(\underline{U}^{\mathrm{m}0})\cdot\underline{n}\cdot\underline{U}^{\mathrm{m}\varepsilon}]\,\mathrm{d}l = -1/2\int_{F_E}\sigma(\underline{U}^{\mathrm{m}0})\cdot\underline{n}\cdot\underline{U}^{\mathrm{m}\varepsilon}\,\mathrm{d}l.$$

The integral along F_E means along the two faces F_E^+ and F_E^-. Although it is an interesting formulation from the mechanical point of view, such an expression is not numerically so easy to handle since it requires the computation of stresses along F_E, i.e. in a region where singularities govern the behaviour. Thus we keep Eq. (10) in which calculations can be performed far from this region. Inserting Eqs. (5) and (6) in Eq. (10) leads to (Leguillon, 1989)

$$\Delta W = k_{\mathrm{m}}^2 K\varepsilon^{2\lambda} + \cdots \tag{11}$$

The following property, consequence of the contour independence of the integral ψ,

$$\psi(r^\lambda\underline{u}, r^\mu\underline{v}) \neq 0 \quad \text{only if } \mu = -\lambda,$$

and the normalization

$$\psi(r^\lambda\underline{u}, r^{-\lambda}\underline{u}^-) = 1,$$

are required to settle Eq. (11). Let us define now the incremental energy release rate G

$$G = \frac{\Delta W}{\varepsilon\ell} = \frac{1}{\ell}(k_{\mathrm{m}}^2 K\varepsilon^{2\lambda-1} + \cdots) \tag{12}$$

Obviously it depends on ε and vanishes or tends to infinity as $\varepsilon \to 0$ according to $\lambda > 1/2$ or $\lambda < 1/2$. In the particular case $\lambda = 1/2$, the leading term in Eq. (12) is independent of the increment and the definition coincides with the usual differential one (in 2D)

$$G = -\frac{\partial W}{\partial \ell},$$

where ℓ is the primary crack length.

As already mentioned, K in Eq. (11) depends on the shape of the perturbation, thus two energy release rates must be computed, one for deflection (index d) and one for penetration (index p)

$$G_{\mathrm{d/p}} = \frac{1}{\ell}(k_{\mathrm{m}}^2 K_{\mathrm{d/p}}\varepsilon_{\mathrm{d/p}}^{2\lambda-1} + \cdots),$$

and the deflection and penetration criteria write respectively (see Eq. (9))

$$\frac{1}{\ell}(k_{\mathrm{m}}^2 K_{\mathrm{d}}\varepsilon_{\mathrm{d}}^{2\lambda-1}) > G_{i\mathrm{c}}, \qquad \frac{1}{\ell}(k_{\mathrm{m}}^2 K_{\mathrm{p}}\varepsilon_{\mathrm{p}}^{2\lambda-1}) > G_{2\mathrm{c}}.$$

Thus, deflection is promoted if

$$\frac{G_{\mathrm{d}}}{G_{\mathrm{p}}} = \frac{K_{\mathrm{d}}}{K_{\mathrm{p}}}\left(\frac{\varepsilon_{\mathrm{d}}}{\varepsilon_{\mathrm{p}}}\right)^{2\lambda-1} > \frac{G_{i\mathrm{c}}}{G_{2\mathrm{c}}}. \tag{13}$$

This form of the He and Hutchinson criterion has been proposed by Leguillon and Sanchez-Palencia (1992). It is a pure local criterion, obviously independent of the applied loads and the geometry of the specimen. It is obtained by assuming $G_{\mathrm{d}} = G_{i\mathrm{c}}$ and $G_{\mathrm{p}} < G_{2\mathrm{c}}$, i.e. deflection can occur while penetration is inhibited, k_{m}^2 is extracted from the equation and inserted in the inequality. This criterion takes a much simplified form with the additional He and Hutchinson assumption $\varepsilon_{\mathrm{d}} = \varepsilon_{\mathrm{p}}$ (see Section 1.1)

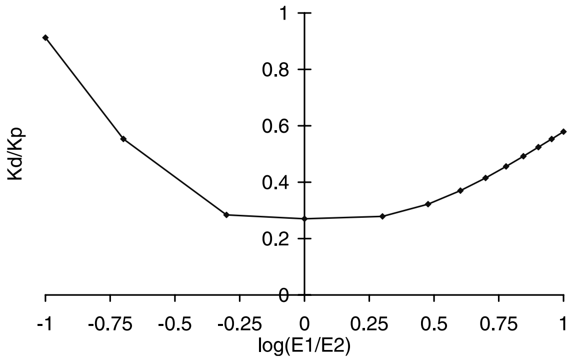

Fig. 6. The deflection/penetration criterion (14) ($\varepsilon_d = \varepsilon_p$) vs. the Young's moduli ratio $E^{(1)}/E^{(2)}$ for $v = 0.3$. The toughnesses ratio G_{ic}/G_{2c} below the line entails deflection.

Table 1
Singularity exponents vs. the Young's moduli ratio $E^{(1)}/E^{(2)}$ for $v = 0.3$ computed from Leguillon and Sanchez-Palencia (1987)

$E^{(1)}/E^{(2)}$	0.1	0.2	0.5	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
λ	0.67	0.63	0.57	0.50	0.42	0.38	0.35	0.32	0.30	0.28	0.27	0.26	0.23

$$\frac{G_d}{G_p} = \frac{K_d}{K_p} > \frac{G_{ic}}{G_{2c}}. \tag{14}$$

Comparisons exhibiting a good agreement between the form (14) of the criterion and He and Hutchinson results have already been proposed in (Leguillon et al., 2000a,b) for different values of the Dundurs mismatch coefficients (Dundurs, 1967), they are not reproduced here. Fig. 6 plots the ratio K_d/K_p vs. $E^{(1)}/E^{(2)}$ for $v = 0.3$. The Young's moduli ratio seems to us a more significant parameter to describe the elastic mismatch between components. The singularity exponents are shown in Table 1.

2. The role of higher-order terms in the asymptotics

2.1. The first non-singular term

There are two second-order terms in the expansion of the solution near the primary crack tip which are homogeneous to r. The first one corresponds to the rigid rotation and is irrelevant. The second one is the generalization to the non-homogeneous case of the well known "T-stress" in homogeneous materials, it is analytically known

$$\underline{U}(x_1, x_2) = r\underline{t}(\varphi). \tag{15}$$

The stress field is constant in each subdomain

$$\begin{cases} \sigma_{11}^{(1)} = \sigma_{11}^{(2)} = s, \\ \sigma_{22}^{(1)} = 0, \quad \sigma_{22}^{(2)} = \sigma_{22}, \\ \sigma_{12}^{(1)} = \sigma_{12}^{(2)} = 0, \end{cases} \tag{16}$$

and the associated displacement field writes

$$\begin{cases} t_1^{(1)}(\varphi) = \frac{(1+\nu)}{E^{(1)}}(1-\nu)s\cos(\varphi), \\ t_2^{(1)}(\varphi) = -\frac{(1+\nu)}{E^{(1)}}\nu s\sin(\varphi), \\ t_1^{(2)}(\varphi) = \frac{(1+\nu)}{E^{(2)}}[(1-\nu)s - \nu\sigma_{22}]\cos(\varphi), \\ t_2^{(2)}(\varphi) = \frac{(1+\nu)}{E^{(2)}}[(1-\nu)\sigma_{22} - \nu s]\sin(\varphi), \end{cases} \tag{17}$$

with

$$\sigma_{22} = \frac{\nu}{(1-\nu)}\frac{E^{(1)} - E^{(2)}}{E^{(1)}}s.$$

It is a non-singular symmetrical mode and depends on a single multiplicative parameter s. Clearly, $\sigma_{22}^{(1)} = \sigma_{22}^{(2)} = 0$ if $E^{(1)} = E^{(2)}$, i.e. if the two materials are identical. This allows to recover the "T-stress" solution $\sigma_{11}^{(1)} = \sigma_{11}^{(2)} = s$ (and vanishing other components).

2.2. The next non-singular term

As will be seen in the forthcoming sections, the next term in the expansion in increasing powers of r is not really necessary to carry out an improved asymptotic analysis involving thermal loadings, but in a first step it must be taken into account to decide which terms are negligible or not. This next term can be calculated (Leguillon and Sanchez-Palencia, 1987) and writes

$$\underline{W}(x_1, x_2) = r^{\zeta}\underline{w}(\varphi),$$

as for the previous terms, there are two modes, one symmetrical and one antisymmetrical, associated with the exponent ζ, but the antisymmetrical one is irrelevant in case of symmetrical loadings and will be omitted here (the intensity factor in the Williams-like series vanishes). The exponent ζ depends as λ on the relative stiffness of the materials, $3/2 < \zeta$ if $E^{(1)} < E^{(2)}$, $1 < \zeta < 3/2$ if $E^{(1)} > E^{(2)}$, the limit $\zeta = 3/2$ corresponds to a crack in a homogeneous material. Moreover if $E^{(1)} < 0.45E^{(2)}$ (about) then $\zeta > 2$ and the term must be neglected otherwise one would have to consider in place the integer term r^2. Indeed, the expansions contains integer powers of r and in addition non-integer ones like $\lambda, \zeta, \ldots$, thus ζ must be compared to 2 before truncation of the series.

2.3. Higher-order matched asymptotics

The matching of the asymptotics must be revisited and becomes more entangled. With two additional terms, the behaviour of the leading terms $\underline{U}^{m0}$ and $\underline{U}^{m1}$ (see Eq. (7)) now reads (with the previous restrictions concerning the last term r^{ζ})

$$\begin{cases} \underline{U}^{m0}(x_1, x_2) = \underline{U}^{m0}(0,0) + k_m r^{\lambda}\underline{u}(\varphi) + T_m r\underline{t}(\varphi) + p_m r^{\zeta}\underline{w}(\varphi) + \cdots, \\ \underline{U}^{m1}(x_1, x_2) = r^{-\lambda}\underline{u}^{-}(\varphi) + \underline{U}^{m1}(0,0) + q_m r^{\lambda}\underline{u}(\varphi) + \cdots, \end{cases} \tag{18}$$

where T_m is the intensity factor of the non-singular mode $r\underline{t}(\varphi)$. In the following, we set $s = 1$, then T_m is the component σ_{11} of the non-singular generalized "T-stress". In the four-point bending test illustrated in Fig. 1, it is numerically checked that $T_m < 0$ if the material 1 is softer than the material 2 and reciprocally $T_m > 0$ if the material 1 is stiffer. The coefficient p_m is the intensity factor of the next non-singular term and q_m is the intensity factor of the singular term in $\underline{U}^{m1}(x_1, x_2)$, it is independent of the applied loads but depends on the geometry of the specimen (see Eq. (8)). As a first consequence of the matching conditions, with a crack increment with length ε either in penetration or deflection, three higher-order terms can be exhibited in the inner expansions

$$\begin{cases} \underline{U}^{m0}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{m0}(0,0) + k_m \varepsilon^\lambda \rho^\lambda \underline{u}(\varphi) + T_m \varepsilon \rho \underline{t}(\varphi) + p_m \varepsilon^\zeta \rho^\zeta \underline{w}(\varphi) + \cdots, \\ \underline{U}^{m\varepsilon}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{m0}(0,0) + k_m \varepsilon^\lambda [\rho^\lambda \underline{u}(\varphi) + \hat{V}^1(y_1, y_2)] \\ \qquad + T_m \varepsilon [\rho \underline{t}(\varphi) + \underline{\hat{V}}^2(y_1, y_2)] + k_m q_m \varepsilon^{3\lambda} [\rho^\lambda \underline{u}(\varphi) + \underline{\hat{V}}^1(y_1, y_2)] \\ \qquad + p_m \varepsilon^\zeta [\rho^\zeta \underline{w}(\varphi) + \underline{\hat{V}}^3(y_1, y_2)] + \cdots \end{cases} \tag{19}$$

As done in Section 1.2 for the first term, the splittings

$$\begin{cases} \underline{V}^2(y_1, y_2) = \rho \underline{t}(\varphi) + \underline{\hat{V}}^2(y_1, y_2), \\ \underline{V}^3(y_1, y_2) = \rho^\zeta \underline{w}(\varphi) + \underline{\hat{V}}^3(y_1, y_2), \end{cases} \tag{20}$$

ensure respectively the second term to behave like $\rho \underline{t}(\varphi)$ and the third one like $\rho^\zeta \underline{w}(\varphi)$ at infinity (new matching conditions).

The entanglement arises in Eq. (19), as a matter of fact the place of the last term ε^ζ in the expansions is not so clear, the exponent 3λ must be compared to ζ:

- If $E^{(1)} < E^{(2)}$ it is numerically checked (Leguillon and Sanchez-Palencia, 1987) that $3\lambda < \zeta$, moreover if $\lambda > 2/3$, i.e. $E^{(1)} < 0.1E^{(2)}$ (about), both 3λ and ζ must be considered as extra terms and neglected (otherwise one would account for another term involving ε^2, see Section 2.2).
- If $E^{(1)} > E^{(2)}$ the same numerical conclusion can be drawn $3\lambda < \zeta$, moreover if $\lambda < 1/3$, i.e. $E^{(1)} > 4.5E^{(2)}$ (about) then $\varepsilon^{3\lambda}$ precedes the ε "T-stress" term itself.
- If $E^{(1)} = E^{(2)}$, $3\lambda = \zeta$, the two last terms of Eq. (19) confound, this will be the topic of Section 5.

Additional terms are now required to describe the behaviour of the complementary terms $\underline{\hat{V}}^j(y_1, y_2)$ $(j = 1, 2, 3)$ at infinity (to compare to Eq. (6))

$$\begin{cases} \underline{\hat{V}}^1(y_1, y_2) \sim K\rho^{-\lambda}\underline{u}^-(\varphi) + H\rho^{-1}\underline{t}^-(\varphi) + P\rho^{-\zeta}\underline{w}^-(\varphi) + \cdots, \\ \underline{\hat{V}}^2(y_1, y_2) \sim K'\rho^{-\lambda}\underline{u}^-(\varphi) + H'\rho^{-1}\underline{t}^-(\varphi) + P'\rho^{-\zeta}\underline{w}^-(\varphi) + \cdots, \qquad \text{as } \rho \to \infty. \\ \underline{\hat{V}}^3(y_1, y_2) \sim K''\rho^{-\lambda}\underline{u}^-(\varphi) + H''\rho^{-1}\underline{t}^-(\varphi) + P''\rho^{-\zeta}\underline{w}^-(\varphi) + \cdots, \end{cases} \tag{21}$$

Here $\rho^{-1}\underline{t}^-(\varphi)$ and $\rho^{-\zeta}\underline{w}^-(\varphi)$ are the dual modes to $\rho\underline{t}(\varphi)$ and $\rho^\zeta\underline{w}(\varphi)$. It is not necessary to proceed to a new matching and calculate complementary terms in the outer expansion to derive the asymptotics of the energy release rate, the contour integral ψ (see Eq. (10) in Section 1.2) can be computed in the inner domain. Using Eqs. (19) and (21) leads obviously to a very awkward expression which is difficult to sort in ascending powers of ε. Then an improved but still simple form of the energy release rate expansion is obtained by a truncation at the order defined by the minimum of λ and $4\lambda - 1$

$$G = \frac{1}{\ell}[k_m^2\, K\varepsilon^{2\lambda-1} + k_m T_m (K' + H)\ \varepsilon^\lambda + \cdots] \tag{22}$$

if $\lambda > 1/3$, or

$$G = \frac{1}{\ell}(k_m^2\, K\varepsilon^{2\lambda-1} + k_m^2 q_m K\varepsilon^{4\lambda-1} + \cdots) \tag{23}$$

if $\lambda < 1/3$. The exponent ζ is no longer involved and the corresponding terms in Eqs. (18)–(21) will be neglected in the following. It is to be pointed out that K, K' and H are concerned with the geometry of the perturbation (deflection or penetration) whereas k_m, T_m and q_m are not. Then, using Eqs. (22) and (23) first for deflection and next for penetration leads to rewrite the deflection criterion as

$$\frac{G_d}{G_p} = \frac{K_d + (K'_d + H_d)\eta_d^m}{K_p + (K'_p + H_p)\eta_p^m}\left(\frac{\varepsilon_d}{\varepsilon_p}\right)^{2\lambda-1} > \frac{G_{ic}}{G_{2c}}, \tag{24}$$

with

$$\eta^{\mathrm{m}}_{\mathrm{d/p}} = \frac{T_{\mathrm{m}}}{k_{\mathrm{m}}} \varepsilon^{1-\lambda}_{\mathrm{d/p}}, \tag{25}$$

if $\lambda > 1/3$, or

$$\frac{G_{\mathrm{d}}}{G_{\mathrm{p}}} = \frac{K_{\mathrm{d}}}{K_{\mathrm{p}}} \frac{1 + q_{\mathrm{m}}\varepsilon_{\mathrm{d}}^{2\lambda}}{1 + q_{\mathrm{m}}\varepsilon_{\mathrm{p}}^{2\lambda}} \left(\frac{\varepsilon_{\mathrm{d}}}{\varepsilon_{\mathrm{p}}}\right)^{2\lambda-1} > \frac{G_{i\mathrm{c}}}{G_{2\mathrm{c}}}, \tag{26}$$

if $\lambda < 1/3$. Very similar parameters to $\eta^{\mathrm{m}}_{\mathrm{d/p}}$ have been introduced by He et al. (1994) in view of analysing the influence of residual thermal stresses. This will be discussed below in Section 4, but emphasis is put already now on the fact that, even with the He and Hutchinson simplification $\varepsilon_{\mathrm{d}} = \varepsilon_{\mathrm{p}}$, the criterion (24) still depends on the increment lengths through $\eta^{\mathrm{m}}_{\mathrm{d/p}} = \eta^{\mathrm{m}}$, whereas Eq. (26) becomes independent of them and writes as the first order criterion (14), although an additional term is used in the expansions. On the other hand, both criteria (24) and (26) are independent of the applied loads because either parameters in Eq. (25) or q_{m} (obviously from Eq. (8)) are independent of g. Indeed, the intensity factors are proportional to g

$$k_{\mathrm{m}} = \kappa_{\mathrm{m}} g, \quad T_{\mathrm{m}} = \tau_{\mathrm{m}} g, \tag{27}$$

where κ_{m} and τ_{m} depend only on the geometry of the structure, and then

$$\eta^{\mathrm{m}}_{\mathrm{d/p}} = \frac{\tau_{\mathrm{m}}}{\kappa_{\mathrm{m}}} \varepsilon^{1-\lambda}_{\mathrm{d/p}}.$$

It is the reason why the criterion takes again the form (24), g^2 is extracted from the equation $G_{\mathrm{d}} = G_{i\mathrm{c}}$ and inserted in $G_{\mathrm{p}} < G_{2\mathrm{c}}$. The improved form of the criterion remains independent of the intensity of the applied loads (see Eq. (13)) but depends now on the geometry of the specimen through the parameters κ_{m} and τ_{m} or q_{m}. It is no longer a local criterion. Fig. 7 shows this criterion when the material 1 is softer than the material 2 (weak singularity) with the simplifying assumption $\varepsilon_{\mathrm{d}} = \varepsilon_{\mathrm{p}}$, for different values of the singularity exponent. For large values of it, it is extremely dependent on the increment length. The next Fig. 8 shows the same criterion when the material 1 is now stiffer than the material 2. It is moderately sensitive to the increment length or even totally insensitive in case of a very strong singularity ($\lambda < 1/3$). In particular, note that the range on the horizontal axis is wider than that of Fig. 7. The non-locality of the criterion implies that these curves depend on the geometry of the specimen and especially on the kind of test which is worked out (the four-point bending test of Fig. 1 here).

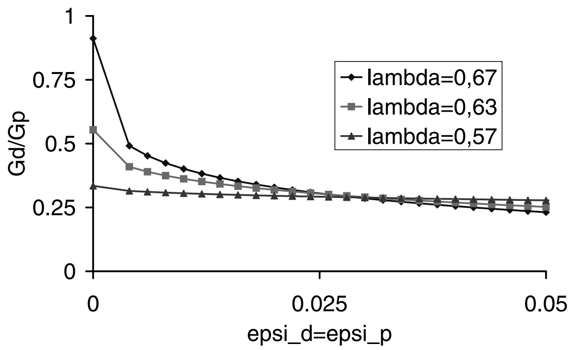

Fig. 7. The deflection/penetration criterion (24) vs. the increment length $\varepsilon_{\mathrm{d}} = \varepsilon_{\mathrm{p}}$, for different elastic contrasts corresponding to a weak singularity ($\lambda > 1/2$). The values at $\varepsilon_{\mathrm{d}} = \varepsilon_{\mathrm{p}} = 0$ correspond to the first-order criterion (14).

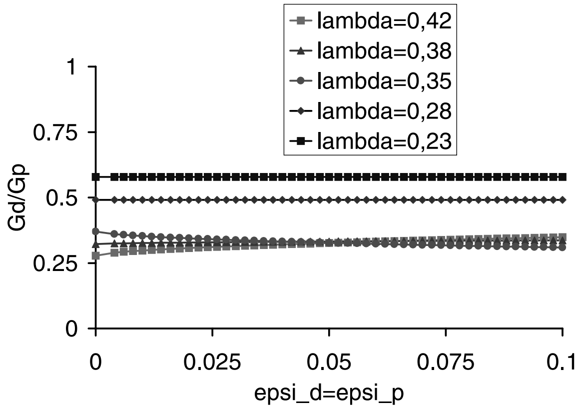

Fig. 8. The deflection/penetration criterion (24) vs. the increment length $\varepsilon_d = \varepsilon_p$, for different elastic contrasts corresponding to a strong singularity ($\lambda < 1/2$). The two horizontal lines in the top corresponds to very strong singularities ($\lambda < 1/3$), they are derived from Eq. (26) and keep constant whatever the increment length.

3. Residual thermal stresses

3.1. Introduction

Considering a uniform temperature change $\Delta\Theta = \Theta$, the thermoelastic constitutive law of an isotropic body is a linear relation between the stress field σ, the strain field $\nabla_s \underline{U}$ (i.e. the symmetrical part of the gradient of the displacement) and Θ, it replaces the second term of Eq. (1).

$$\sigma = C \cdot (\nabla_s \underline{U} - \alpha I \Theta) = C \cdot \nabla_s \underline{U} - \beta I \Theta, \tag{28}$$

where α (K^{-1}) is the thermal expansion coefficient (I is the identity tensor, the expansion is assumed isotropic as well). Coefficient β (MPa K^{-1}) allows to write down the constitutive law in a slightly different way which will be useful in the forthcoming developments. In plane strain elasticity, these coefficients are linked together, using the Poisson's ratio ν and the Young's modulus E, by the relation

$$\alpha = \frac{1+\nu}{E}(1-2\nu)\beta.$$

Equilibrium equation and boundary conditions (1) (first term) and (1) (third to fifth terms) must be added to Eq. (28) to complete the set of equations of the thermomechanical problem on a domain Ω with boundary $\partial\Omega$. The problem splits into two parts, a pure mechanical one accounting for the prescribed loads and solution to Eq. (1), and a thermal contribution (index θ) satisfying

$$\sigma^\theta = C \cdot \left(\nabla_s \underline{U}^\theta - \alpha I \Theta\right), \quad \sigma^\theta \cdot \underline{n} = 0 \quad \text{on } \partial\Omega.$$

In a bimaterial $\Omega = \Omega^1 \cup \Omega^2$, the displacement field $\underline{U}^\theta = \underline{U}^{\theta(k)}$ in $\Omega^{(k)}$ $(k = 1, 2)$ is proportional to Θ and must be solution to the following boundary value problem

$$\begin{cases} -\nabla.\tilde{\sigma} = 0 \quad \text{in } \Omega, \\ \tilde{\sigma} = C \cdot \nabla_s \underline{U}^\theta \quad \text{in } \Omega, \\ \tilde{\sigma}_{12} = 0, \qquad U_2^\theta = 0, \quad \text{on } \Gamma_s, \\ [\![\tilde{\sigma}]\!] \cdot \underline{n} = [\![\beta]\!] \Theta \underline{n} \quad \text{on } \Gamma, \\ \tilde{\sigma}^{(k)} \cdot \underline{n} = \beta^{(k)} \Theta \underline{n} \quad \text{elsewhere on } \partial\Omega \cap \Omega^{(k)}, \end{cases} \tag{29}$$

where the brackets $[\![\,]\!]$ denote a jump through the interface Γ located between $\Omega^{(1)}$ and $\Omega^{(2)}$

$$[\![\tilde{\sigma}]\!] = \tilde{\sigma}^{(2)} - \tilde{\sigma}^{(1)}, \quad [\![\beta]\!] = \beta^{(2)} - \beta^{(1)}.$$

In the most general case, problem (29) must be numerically solved by a finite element method for instance. For our purpose, we need only a description of the solution in the vicinity of the crack tip, disregarding conditions on remote boundaries. Thus we have to solve Eq. (29) (first to third terms) with Eq. (29) (fifth term) on the two faces F^+ and F^- of the primary crack and Eq. (29) (fourth term) on the interface Γ. The stress components fulfil constant boundary conditions, as a consequence the solutions (of course there is no uniqueness) are consistent with a second-order term in the asymptotics (see Section 2.1) which reads in polar coordinates r and φ

$$\underline{U}_{\rm d}(x_1, x_2) = \Theta r \underline{u}_{\rm d}(\varphi). \tag{30}$$

The explanation for the index d will be given in Section 3.2. One simple solution to this problem writes

$$\begin{cases} \tilde{\sigma}^{(1)}_{\rm d11} = \tilde{\sigma}^{(1)}_{\rm d22} = \Theta\beta^{(1)}, \\ \tilde{\sigma}^{(2)}_{\rm d11} = \Theta\beta^{(2)}, \qquad \tilde{\sigma}^{(2)}_{\rm d22} = \Theta\tilde{s}, \\ \tilde{\sigma}^{(1)}_{\rm d12} = \tilde{\sigma}^{(2)}_{\rm d12} = 0, \end{cases} \tag{31}$$

where $\tilde{\sigma}^{(k)}_{{\rm d}ij}$ denotes the component ij of Eq. (29) (second term) in the subdomain k and where the constant $\tilde{s}$ is defined below Eq. (33). Then the displacement field can be derived

$$\begin{cases} u^{(1)}_{\rm d1}(\varphi) = \frac{(1+\nu)}{E^{(1)}}(1-2\nu)\beta^{(1)}\cos(\varphi) = \alpha^{(1)}\cos(\varphi), \\ u^{(1)}_{\rm d2}(\varphi) = \frac{(1+\nu)}{E^{(1)}}(1-2\nu)\beta^{(1)}\sin(\varphi) = \alpha^{(1)}\sin(\varphi), \\ u^{(2)}_{\rm d1}(\varphi) = \frac{(1+\nu)}{E^{(2)}}[(1-\nu)\beta^{(2)} - \nu\tilde{s}]\cos(\varphi), \\ u^{(2)}_{\rm d2}(\varphi) = \frac{(1+\nu)}{E^{(2)}}[(1-\nu)\tilde{s} - \nu\beta^{(2)}]\sin(\varphi), \end{cases} \tag{32}$$

with

$$\frac{1-\nu}{E^{(2)}}\tilde{s} = \frac{1-2\nu}{E^{(1)}}\beta^{(1)} + \frac{\nu}{E^{(2)}}\beta^{(2)}. \tag{33}$$

The stress field associated with this solution is

$$\begin{cases} \sigma^{(1)}_{\rm d11} = \sigma^{(1)}_{\rm d22} = \sigma^{(2)}_{\rm d11} = \sigma^{(1)}_{\rm d12} = \sigma^{(2)}_{\rm d12} = 0, \\ \sigma^{(2)}_{\rm d22} = (\tilde{s} - \beta^{(2)})\Theta = \frac{1-2\nu}{1-\nu}\left(\frac{E^{(2)}}{E^{(1)}}\beta^{(1)} - \beta^{(2)}\right)\Theta = \frac{-1}{1-\nu}\,\frac{E^{(2)}}{1+\nu}[\![\alpha]\!]\Theta, \end{cases} \tag{34}$$

which brings into evidence the role of the mismatch $[\![\alpha]\!] = \alpha^{(2)} - \alpha^{(1)}$ in the thermal expansion coefficients. It is clear that such a vector (30) is defined up to a solution (15) of the homogeneous problem (i.e. satisfying homogeneous boundary conditions near the crack tip).

3.2. Asymptotics of residual thermal stresses

We consider in this section a pure thermal problem. The unperturbed and perturbed solutions now read respectively $\underline{U}^{\theta 0}$ (solution to Eq. (29)) and $\underline{U}^{\theta\varepsilon}$. In a first step, the unperturbed solution is splitted into an homogeneous term (i.e. fulfiling homogeneous boundary conditions in the vicinity of the primary crack tip, index h) and a particular solution which is taken to be $\underline{U}_{\rm d}$ (30)

$$\underline{U}^{\theta 0}(x_1, x_2) = \underline{U}^{\rm h0}_{\rm d}(x_1, x_2) + \underline{U}_{\rm d}(x_1, x_2). \tag{35}$$

Of course, the so-called homogeneous term depends on the choice of the particular solution. Then the previous results concerning asymptotics can be applied to $\underline{U}^{\rm h0}_{\rm d}$ with the truncation defined in Section 2.3

$$\begin{cases} \underline{U}_{\mathrm{d}}^{\mathrm{h0}}(x_1,x_2) = \underline{U}_{\mathrm{d}}^{\mathrm{h0}}(0,0) + k_{\theta\mathrm{d}} r^{\lambda}\underline{u}(\varphi) + T_{\theta\mathrm{d}} r\underline{t}(\varphi) + \cdots, \\ \underline{U}^{\theta 1}(x_1,x_2) = r^{-\lambda}\underline{u}^{-}(\varphi) + \underline{U}^{\theta 1}(0,0) + q_{\theta} r^{\lambda}\underline{u}(\varphi) + \cdots, \end{cases} \tag{36}$$

where q_θ is independent of Θ and $[\![\alpha]\!]$. By analogy, $k_{\theta\mathrm{d}}$ can be baptised TSIF (Meyer and Schmauder, 1992). Moreover from Eq. (8) it comes $\underline{U}^{\mathrm{m}1} = \underline{U}^{\theta 1}$ and then

$$q_\theta = q_{\mathrm{m}} = q. \tag{37}$$

The perturbed solution $\underline{U}_{\mathrm{d}}^{\theta\varepsilon}$ (deflection) or $\underline{U}_{\mathrm{p}}^{\theta\varepsilon}$ (penetration) has to satisfy additional boundary conditions on the faces F_E^+ and F_E^- of the crack extension. They depend on the mechanism, for deflection it is

$$\begin{cases} \tilde{\sigma}(\underline{U}_{\mathrm{d}}^{\theta\varepsilon}) \cdot \underline{n} = \beta^{(1)}\Theta\underline{n} \quad \text{on } F_E^+, \\ \tilde{\sigma}(\underline{U}_{\mathrm{d}}^{\theta\varepsilon}) \cdot \underline{n} = \beta^{(2)}\Theta\underline{n} \quad \text{on } F_E^-, \end{cases} \tag{38}$$

and for penetration

$$\tilde{\sigma}(\underline{U}_{\mathrm{p}}^{\theta\varepsilon}) \cdot \underline{n} = \beta^{(2)}\Theta\underline{n} \quad \text{on } F_E^+ \text{ and } F_E^-, \tag{39}$$

with

$$\tilde{\sigma}(\underline{U}_{\mathrm{d/p}}^{\theta\varepsilon}) = C \cdot \nabla_{\mathrm{s}}\underline{U}_{\mathrm{d/p}}^{\theta\varepsilon} \quad \text{and} \quad \sigma(\underline{U}_{\mathrm{d/p}}^{\theta\varepsilon}) = \tilde{\sigma}(\underline{U}_{\mathrm{d/p}}^{\theta\varepsilon}) - \beta I\Theta.$$

It is clear that the particular solution $r\underline{u}_{\mathrm{d}}(\varphi)$ (32) fulfils Eq. (38) but not Eq. (39) (this is the reason for the index d). Nevertheless, it is not unique and we shall now define another particular solution

$$\underline{U}_{\mathrm{p}}(x_1,x_2) = \Theta\ r\underline{u}_{\mathrm{p}}(\varphi),$$

which satisfies Eq. (39) instead of Eq. (38), using a combination of Eqs. (17) and (32) by adjusting the parameter s. Taking

$$s = \frac{1-2v}{v}\ \frac{E^{(1)}\beta^{(2)} - E^{(2)}\beta^{(1)}}{E^{(1)} - E^{(2)}} = \frac{E^{(1)}}{v(E^{(1)} - E^{(2)})}\ \frac{E^{(2)}}{1+v}[\![\alpha]\!], \tag{40}$$

leads to

$$\begin{cases} \tilde{\sigma}_{\mathrm{p}11}^{(1)} = \Theta(s + \beta^{(1)}), \quad \tilde{\sigma}_{\mathrm{p}11}^{(2)} = \Theta(s + \beta^{(2)}), \\ \tilde{\sigma}_{\mathrm{p}22}^{(1)} = \Theta\beta^{(1)}, \quad \tilde{\sigma}_{\mathrm{p}22}^{(2)} = \Theta\beta^{(2)}, \\ \tilde{\sigma}_{\mathrm{p}12}^{(1)} = \tilde{\sigma}_{\mathrm{p}12}^{(2)} = 0, \end{cases}$$

and then

$$\begin{cases} \sigma_{\mathrm{p}22}^{(1)} = \sigma_{\mathrm{p}22}^{(2)} = \sigma_{\mathrm{p}12}^{(1)} = \sigma_{\mathrm{p}12}^{(2)} = 0, \\ \sigma_{\mathrm{p}11}^{(1)} = \sigma_{\mathrm{p}11}^{(2)} = s\Theta. \end{cases}$$

We emphasize on the following point: *it is impossible to carry out the above calculation of the new particular solution* $\underline{U}_{\mathrm{p}}$ *if the two materials have identical elastic properties and different expansion coefficients* (see Section 5). Of course, if both are identical, there is no problem because residual stresses are absent.

The associated displacement field $\underline{U}_{\mathrm{p}}(x_1,x_2)$ derives from Eqs. (15)–(17) and (30)–(32) with Eq. (40). The analogous to Eq. (35) is

$$\underline{U}^{\theta 0}(x_1,x_2) = \underline{U}_{\mathrm{p}}^{\mathrm{h0}}(x_1,x_2) + \underline{U}_{\mathrm{p}}(x_1,x_2),$$

and Eq. (36) (first term) rewrites (while Eq. (36) (second term) still holds)

$$\underline{U}_{\mathrm{p}}^{\mathrm{h0}}(x_1,x_2) = \underline{U}_{\mathrm{p}}^{\mathrm{h0}}(0,0) + k_{\theta\mathrm{p}} r^{\lambda}\underline{u}(\varphi) + T_{\theta\mathrm{p}} r\underline{t}(\varphi) + \cdots$$

Moreover, the difference between $\underline{U}_{\rm d}^{\rm h0}$ and $\underline{U}_{\rm p}^{\rm h0}$ is the non-singular term (15) (i.e. a term homogeneous to r), then

$$k_{\theta{\rm d}} = k_{\theta{\rm p}} = k_\theta, \quad T_{\theta{\rm p}} = T_{\theta{\rm d}} + s\Theta,$$

(see Eq. (40) and the remark following Eq. (18)). Depending on the material mismatch, the thermal loading can cause either a crack opening or closure. In case of closure, with the present boundary conditions it leads to $k_\theta < 0$, i.e. the crack faces overlap. But, as will be discussed in Section 4, this solution must be kept in this form and is significant when combined loadings (i.e. mechanical and thermal) are under considerations.

Otherwise, in case of a single thermal loading, either the condition $k_\theta > 0$ holds and a fracture analysis can be carried out, or the closure condition $[\![\underline{U}^\theta]\!] \cdot \underline{n} = 0 \Rightarrow k_\theta = 0$ on the crack faces, makes such an analysis unnecessary. The homogeneous parts of the unperturbed solution $\underline{U}_{\rm d/p}^{\rm h0}$ are solution to the following boundary value problems

$$\begin{cases} -\nabla \cdot \sigma_{\rm d} = 0 \quad \text{in } \Omega, \\ \sigma_{\rm d} = C \cdot \nabla_{\rm s} \underline{U}_{\rm d}^{\rm h0} \quad \text{in } \Omega, \\ \sigma_{\rm d} \cdot \underline{n} = \frac{1}{1-v}\frac{E^{(2)}}{1+v}[\![\alpha]\!]\Theta\underline{n} \quad \text{on } \Gamma_{\rm d}, \\ [\![\sigma_{\rm d}]\!] \cdot \underline{n} = 0 \quad \text{on } \Gamma, \\ \sigma_{\rm d12} = 0, \qquad U_{\rm d2}^{\rm h0} = 0 \quad \text{on } \Gamma_{\rm s}, \\ \sigma_{\rm d} \cdot \underline{n} = 0 \text{ elsewhere on } \partial\Omega, \end{cases} \tag{41}$$

where $\Gamma_{\rm d}$ is the upper part of the boundary in $\Omega^{(2)}$ (see Fig. 9),

$$\begin{cases} -\nabla \cdot \sigma_{\rm p} = 0 \quad \text{in } \Omega, \\ \sigma_{\rm p} = C \cdot \nabla_{\rm s} \underline{U}_{\rm p}^{\rm h0} \quad \text{in } \Omega, \\ \sigma_{\rm p} \cdot \underline{n} = -\Theta s\underline{n} \quad \text{on } \Gamma_{\rm p}, \\ [\![\sigma_{\rm p}]\!] \cdot \underline{n} = 0 \quad \text{on } \Gamma, \\ \sigma_{\rm p12} = 0, \qquad U_{\rm p2}^{\rm h0} = 0 \quad \text{on } \Gamma_{\rm s}, \\ \sigma_{\rm p} \cdot \underline{n} = 0 \text{ elsewhere on } \partial\Omega, \end{cases} \tag{42}$$

where $\Gamma_{\rm p}$ is made of the right part in the boundary of $\Omega^{(1)}$ and the left one in $\Omega^{(2)}$ (see Fig. 9).

Problems (41) and (42) show that these solutions are linear functions of the temperature change Θ and of the expansion coefficients mismatch $[\![\alpha]\!]$ (see Eq. (40)). As a consequence the intensity factors themselves are linear functions of these parameters

$$k_\theta = \kappa_\theta \frac{E^{(2)}}{1+v}\Theta[\![\alpha]\!], \quad T_{\theta{\rm d/p}} = \tau_{\theta{\rm d/p}} \frac{E^{(2)}}{1+v}\Theta[\![\alpha]\!], \tag{43}$$

where κ_θ and $\tau_{\theta{\rm d/p}}$ depend only on the geometry of the specimen in problems (41) and (42). The coefficient $E^{(2)}/(1+v)$ has been introduced for homogeneity reasons (see Eqs. (48) and (49)).

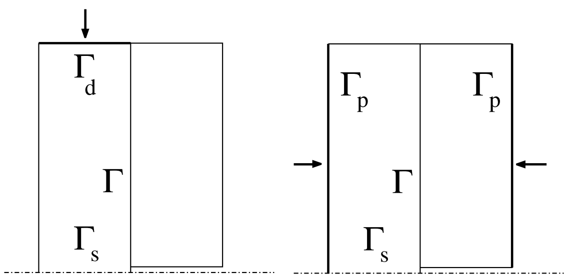

Fig. 9. The equivalent thermal/mechanical problems.

Without new difficulties, the associated inner expansions are derived

$$\begin{cases} \underline{U}^{\theta 0}_{\mathrm{d/p}}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{\mathrm{h}0}_{\mathrm{d/p}}(0,0) + k_\theta \varepsilon^\lambda \rho^\lambda \underline{u}(\varphi) + T_{\theta\mathrm{d/p}}\varepsilon\rho\underline{t}(\varphi) + \Theta\varepsilon\rho\underline{u}_{\mathrm{d/p}}(\varphi) + \cdots, \\ \underline{U}^{\theta\varepsilon}_{\mathrm{d/p}}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{\mathrm{h}0}_{\mathrm{d/p}}(0,0) + k_\theta \varepsilon^\lambda [\rho^\lambda \underline{u}(\varphi) + \underline{\hat{V}}^1_{\mathrm{d/p}}(y_1, y_2)] + T_{\theta\mathrm{d/p}}\varepsilon[\rho\underline{t}(\varphi) + \underline{\hat{V}}^2_{\mathrm{d/p}}(y_1,y_2)] \\ \qquad + \Theta\varepsilon\rho\underline{u}_{\mathrm{d/p}}(\varphi) + \cdots \end{cases} \quad (44)$$

Here, ε is written in place of $\varepsilon_{\mathrm{d/p}}$ for simplicity. The main feature is that an additional thermal term occurs at the same order ε than the generalized "T-stress".

3.3. Energy release rate for thermal loadings

Relation (10) expresses the amount of energy released during the fracture process as a contour integral. Such an expression, based on homogeneous boundary conditions near the crack tip (crack faces are stress free and there is no jump through the interface Γ), is no longer valid for $\underline{U}^{\theta 0}$ and $\underline{U}^{\theta\varepsilon}$. As a starting point, we use the definition of the stored energy proposed by Nairn (1997) to define ΔW

$$\Delta W = 1/2 \int_\Omega \sigma(\underline{U}^{\theta 0}) \cdot (\nabla_{\mathrm{s}} \underline{U}^{\theta 0} - \alpha I \Theta)\,\mathrm{d}x - 1/2 \int_{\Omega^\varepsilon} \sigma(\underline{U}^{\theta\varepsilon}) \cdot (\nabla_{\mathrm{s}} \underline{U}^{\theta\varepsilon} - \alpha I \Theta)\,\mathrm{d}x.$$

As already done, this stored energy change can be written as a work on the crack extension faces

$$\Delta W = -1/2 \int_{F_E} \sigma(\underline{U}^{\theta 0}) \cdot \underline{n} \cdot \underline{U}^{\theta\varepsilon}\,\mathrm{d}l = 1/2 \int_{F_E} [\sigma(\underline{U}^{\theta\varepsilon}) \cdot \underline{n} \cdot \underline{U}^{\theta 0} - \sigma(\underline{U}^{\theta 0}) \cdot \underline{n} \cdot \underline{U}^{\theta\varepsilon}]\,\mathrm{d}l, \quad (45)$$

where the integral along the extension F_E means along $F_E^+ \cup F_E^-$.

In order to get, as in Section 1.2, a contour integral ψ computed far from this region and independent of the selected line, Eq. (45) is rewritten

$$\Delta W_{\mathrm{d/p}} = -1/2 \int_{F_E} \tilde{\sigma}(\underline{U}^{\mathrm{h}0}_{\mathrm{d/p}}) \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}\,\mathrm{d}l - 1/2 \int_{F_E} [\tilde{\sigma}(\underline{U}_{\mathrm{d/p}}) - \beta I \Theta] \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}\,\mathrm{d}l, \quad (46)$$

with

$$\underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}(x_1, x_2) = \underline{U}^{\theta\varepsilon}_{\mathrm{d/p}}(x_1, x_2) - \underline{U}_{\mathrm{d/p}}(x_1, x_2).$$

With the judicious choice of $\underline{U}_{\mathrm{d/p}}$ made previously the second term in the right hand side of Eq. (46) vanishes and moreover

$$\tilde{\sigma}(\underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}) \cdot \underline{n} = 0 \quad \text{on } F_E.$$

Recalling that the index h holds for homogeneous (i.e. homogeneous boundary conditions in the vicinity of the primary crack tip), Eq. (46) becomes

$$\begin{aligned} \Delta W_{\mathrm{d/p}} &= 1/2 \int_{F_E} [\tilde{\sigma}(\underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}) \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}0}_{\mathrm{d/p}} - \tilde{\sigma}(\underline{U}^{\mathrm{h}0}_{\mathrm{d/p}}) \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}]\,\mathrm{d}l \\ &= 1/2 \int_C [\tilde{\sigma}(\underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}) \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}0}_{\mathrm{d/p}} - \tilde{\sigma}(\underline{U}^{\mathrm{h}0}_{\mathrm{d/p}}) \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}]\,\mathrm{d}l = \tilde{\psi}(\underline{U}^{\mathrm{h}\varepsilon}_{\mathrm{d/p}}, \underline{U}^{\mathrm{h}0}_{\mathrm{d/p}}). \end{aligned}$$

Then the expression of the energy release rate is similar to Eqs. (22) and (23), the index θ replacing m. Coefficients K, K' and H are unchanged and it is important to note that at the first-order, the deflection criterion (13) is not altered by the residual stresses. At the second-order, the criterion writes like (24)–(26). Moreover, since $q_\theta = q_{\mathrm{m}} = q$ in the present case Eq. (37), the second form of the criterion (26) remains unchanged whatever the kind of loading.

As already mentioned k_θ and $T_{\theta d/p}$ depend on Θ and on the expansion coefficients mismatch (see Eq. (43)). Thus, in case of thermal loading, the criterion (24) is independent of the temperature change Θ and the expansion coefficient mismatch $[\![\alpha]\!]$ since from Eq. (43)

$$\eta^{\theta}_{d/p} = \frac{\tau_{\theta d/p}}{\kappa_\theta}\varepsilon^{1-\lambda}_{d/p}.$$

This independence holds true obviously for Eq. (26) from the properties of q_θ (see Eq. (36)). Once again $\Theta^2[\![\alpha]\!]^2$ can be extracted from $G_d = G_{ic}$ and inserted in $G_p < G_{2c}$ to get the particular form (24) of the criterion which is non-local. As observed in Figs. 10 and 11, the criterion depends in both cases (weak and strong singularities) on the increment length $\varepsilon_d = \varepsilon_p$. This is mainly due to the strong influence of the "T-stress "intensity factor $T_{\theta d/p}$ in the ratio (26). The range on the horizontal axis is now quite narrow. Figs. 10 and 11 are plotted for $k_\theta > 0$, as already noted, the other case is meaningless since the primary crack is closed.

It is numerically checked (Table 2) that if $[\![\alpha]\!] < 0$, i.e. $\alpha^{(1)} > \alpha^{(2)}$, then both interface and material 2 are in traction ($\Theta < 0$ during a cooling process) in the vicinity of the primary crack tip. This result was not

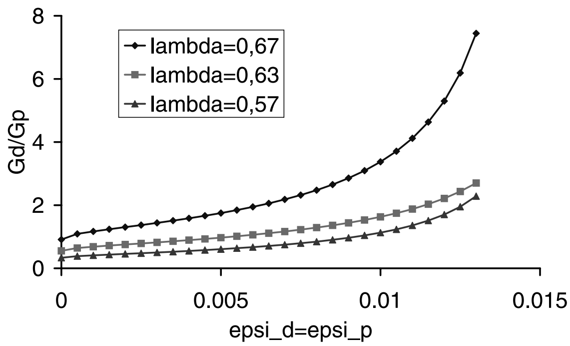

Fig. 10. The deflection/penetration criterion (14) vs. the increment length $\varepsilon_d = \varepsilon_p$, for different elastic contrasts corresponding to a weak singularity ($\lambda > 1/2$). The criterion is independent of the thermal amplitude Θ and the expansion coefficients mismatch $[\![\alpha]\!]$, provided $k_\theta > 0$.

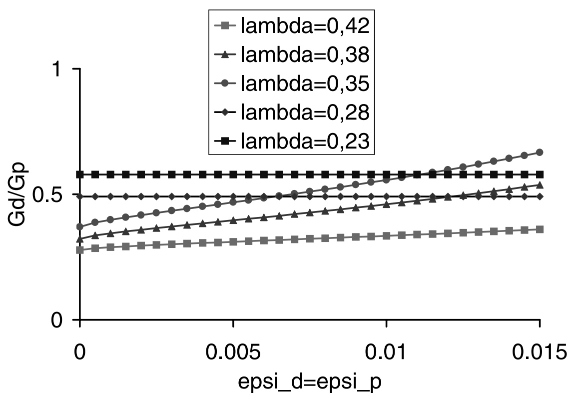

Fig. 11. The deflection/penetration criterion (24) vs. the increment length $\varepsilon_d = \varepsilon_p$, for different elastic contrasts corresponding to a strong singularity ($\lambda < 1/2$). As in Fig. 10 it is independent of Θ and $[\![\alpha]\!]$.

Table 2
The dimensionless traction $\bar{\sigma}_{ii} = ((1+\nu)/E^{(2)})(\sigma_{ii}/[\![\alpha]\!]\Theta)(i=1,2)$ on the interface and ahead of the crack tip for two different elastic contrasts

$E^{(1)}/E^{(2)}$	λ	$\bar{\sigma}_{11}$ (interface)	$\bar{\sigma}_{22}$ (material 2)
0.2	0.63	0.8	1.2
3.0	0.38	0.5	2.6

predictable, the local stress field is governed by the crack tip singularity. On the other hand, the remote stress field is moderate and, as expected, the material having the larger expansion coefficient is in traction during the cooling process. It is a consequence of the geometry and of the stress free boundary conditions, the specimen deforms so as to reduce the residual stresses. As a matter of fact, if $[\![\alpha]\!] < 0$, the near stress field ahead of the crack tip is tensile while the remote one is compressive. Such an unexpected residual stresses redistribution near a crack tip has already been observed by Autesserre (1995) in a quite different situation.

4. Combination of mechanical and thermal loadings, a criterion for crack deflection

In the previous sections, the residual stresses are computed as the result of an equivalent thermal loading acting on the structure. If a mechanical loading (1) is superimposed, the actual intensity factors are the sum of the two contributions

$$k = k_{\mathrm{m}} + k_{\theta}, \qquad T_{\mathrm{d/p}} = T_{\mathrm{m}} + T_{\theta\mathrm{d/p}}, \tag{47}$$

where the indices m (Section 2) and θ (Section 3) hold respectively for mechanical and thermal. Of course the opening condition involves the actual intensity factor and reads $k = k_{\mathrm{m}} + k_{\theta} > 0$ whatever the sign of k_{θ} (see Section 3.2).

With Eq. (37) the energy release rate still reads like (22) and (23) without index (see Eq. (47)). In the second case it still gives rise to a criterion independent of the applied loads (26). But it is no longer possible to derive without care a criterion independent of the applied loads in the first case. From Eqs. (27), (43) and (47) it becomes

$$\begin{cases} k = g\left(\kappa_{\mathrm{m}} + \kappa_{\theta}\frac{E^{(2)}}{1+\nu}\frac{\Theta[\![\alpha]\!]}{g}\right), \\ T_{\mathrm{d/p}} = g\left(\tau_{\mathrm{m}} + \tau_{\theta\mathrm{d/p}}\frac{E^{(2)}}{1+\nu}\frac{\Theta[\![\alpha]\!]}{g}\right). \end{cases} \tag{48}$$

As already done, we consider that deflection occurs, i.e. $G_{\mathrm{d}} = G_{i\mathrm{c}}$ while penetration is inhibited $G_{\mathrm{p}} < G_{2\mathrm{c}}$, but it is no longer possible to proceed as before. Let g_{c} denotes the critical load such that deflection occurs, we set the dimensionless parameter

$$\xi_{\mathrm{c}} = \frac{E^{(2)}}{1+\nu}\frac{\Theta[\![\alpha]\!]}{g_{\mathrm{c}}}. \tag{49}$$

It measures the relative importance of thermal stresses compared to mechanical ones at onset of the mechanism under consideration.

Then, using the above mentioned equality to extract $(g_{\mathrm{c}})^2$ and inserting in the inequality leads to the usual form of the criterion (24) with

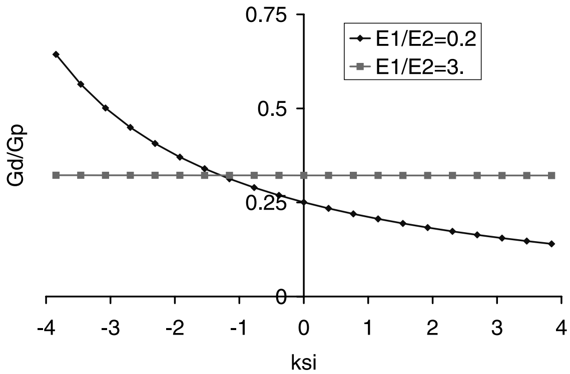

Fig. 12. The influence of residual thermal stresses on the deflection/penetration criterion for a specimen under combined mechanical and thermal loadings. The vertical axis corresponds to the absence of residual stresses, on the right side $\alpha^{(2)} < \alpha^{(1)}$ and the deflection trend is lowered by the residual stresses.

$$\eta_{\mathrm{d/p}} = \frac{T_{\mathrm{d/p}}}{k}\varepsilon_{\mathrm{d/p}}^{1-\lambda} = \frac{\tau_{\mathrm{m}} + \tau_{\theta\mathrm{d/p}}\xi_{\mathrm{c}}}{\kappa_{\mathrm{m}} + \kappa_{\theta}\xi_{\mathrm{c}}}\varepsilon_{\mathrm{d/p}}^{1-\lambda}. \tag{50}$$

The criterion now depends on the ratio ξ_{c}, it makes an explicit reference to the processing temperature Θ and the critical load g_{c}, which is an implicit function of $G_{i\mathrm{c}}$ making Eq. (24) a little bit illusory.

Fig. 12 shows the trends of residual stresses effects. If material 2 is stiffer than material 1 and if $\xi_{\mathrm{c}} < 0$, i.e. $\alpha^{(2)} > \alpha^{(1)}$, then the residual thermal stresses tend to promote deflection. Inversely, deflection is not favoured if $\alpha^{(2)} < \alpha^{(1)}$. Moreover, it is clear in Fig. 12 that the criterion is almost insensitive to residual stresses if material 1 is stiffer than material 2.

Parameters like Eq. (50) have been introduced by He et al. (1994) in their analysis. With a slightly different reasoning, they get an analogous relation to Eq. (24). However, from our point of view, with the present notations their parameters read

$$\eta_{\mathrm{d/p}}^{\mathrm{HH}} = \frac{T_{\theta\mathrm{d/p}}}{k}\varepsilon_{\mathrm{d/p}}^{1-\lambda} \tag{51}$$

(He et al., 1994, footnote on p. 3445). As a matter of fact, there is no reason to neglect T_{m} when it is compared to $T_{\theta\mathrm{d}}$ or $T_{\theta\mathrm{p}}$. Moreover, the criterion is written under the form (24) with an additional squared term $(\eta_{\mathrm{d/p}}^{\mathrm{HH}})^2$ but it has been shown that some preceding terms can exist in the expansions (Section 2.3). Finally, no reference is made to the critical value k_{c} of k corresponding to g_{c} (or even to g_{c} itself), the analysis is simply carried out for different values of the parameters (51).

5. No elastic contrast, a particular case

As mentioned in Section 3.2, in case of penetration, if $E^{(1)} = E^{(2)} = E$ it is not possible (see Eq. (40)) to exhibit a particular solution $\underline{U}_{\mathrm{p}}$ allowing simplifications in the calculation of the energy release rate by a contour integral, $\underline{U}_{\mathrm{d}}$ must be used in both cases. Thus the previous reasoning remains valid in case of deflection and we focus our attention on the penetration event with a pure thermal loading. Of course, the first consequence of the absence of elastic contrast is $\lambda = 1/2$, then $\sqrt{r}\underline{u}(\varphi)$ is the classical mode I. Intensity factors k_{m} and k_{θ} are the usual SIF and TSIF (Meyer and Schmauder, 1992).

The "T-stress" writes

$$\begin{cases} \sigma_{11}^{(1)} = \sigma_{11}^{(2)} = 1, \\ \sigma_{22}^{(1)} = \sigma_{22}^{(2)} = \sigma_{12}^{(1)} = \sigma_{12}^{(2)} = 0, \end{cases} \tag{52}$$

with the associated displacement field

$$\begin{cases} t_1^{(1)}(\varphi) = t_1^{(2)}(\varphi) = \frac{(1+\nu)}{E}(1-\nu)\cos(\varphi), \\ t_2^{(1)}(\varphi) = t_2^{(2)}(\varphi) = -\frac{(1+\nu)}{E}\ \nu\sin(\varphi). \end{cases} \tag{53}$$

For the particular solution $\underline{U}_{\mathrm{d}}(x_1, x_2) = \Theta r \underline{u}_{\mathrm{d}}(\varphi)$, Eq. (31) remains unchanged with a simplified form for $\tilde{s}$

$$\tilde{s} = \frac{1}{(1-\nu)} \frac{E}{(1+\nu)} \left(\alpha^{(1)} + \frac{\nu}{1-2\nu} \alpha^{(2)} \right).$$

The displacement field (analogous to Eq. (32)) writes

$$\begin{cases} u_{\mathrm{d}1}^{(1)}(\varphi) = \alpha^{(1)} \cos(\varphi), \\ u_{\mathrm{d}2}^{(1)}(\varphi) = \alpha^{(1)} \sin(\varphi), \\ u_{\mathrm{d}1}^{(2)}(\varphi) = \frac{1}{1-\nu}\left(\alpha^{(2)} - \nu\alpha^{(1)}\right)\cos(\varphi), \\ u_{\mathrm{d}2}^{(2)}(\varphi) = \alpha^{(1)} \sin(\varphi), \end{cases}$$

and the associated stress field still reads Eq. (34)

$$\begin{cases} \sigma_{\mathrm{d}11}^{(1)} = \sigma_{\mathrm{d}22}^{(1)} = \sigma_{\mathrm{d}11}^{(2)} = \sigma_{\mathrm{d}12}^{(1)} = \sigma_{\mathrm{d}12}^{(2)} = 0, \\ \sigma_{\mathrm{d}22}^{(2)} = \frac{1}{1-\nu}\frac{E}{1+\nu}[\![\alpha]\!]\Theta. \end{cases} \tag{54}$$

The inner expansions (44) now write

$$\begin{cases} \underline{U}_{\mathrm{p}}^{\theta 0}(\varepsilon y_1, \varepsilon y_2) = \underline{U}_{\mathrm{d}}^{\mathrm{h}0}(0,0) + k_\theta\sqrt{\varepsilon}\ \sqrt{\rho}\underline{u}(\varphi) + T_{\theta\mathrm{d}}\ \varepsilon\rho\underline{t}(\varphi) + \Theta\varepsilon\rho\underline{u}_{\mathrm{d}}(\varphi) + \cdots, \\ \underline{U}_{\mathrm{p}}^{\theta\varepsilon}(\varepsilon y_1, \varepsilon y_2) = \underline{U}_{\mathrm{d}}^{\mathrm{h}0}(0,0) + k_\theta\sqrt{\varepsilon}\ [\sqrt{\rho}\underline{u}(\varphi) + \underline{\hat{V}}_{\mathrm{p}}^{1}(y_1,y_2)] + T_{\theta\mathrm{d}}\ \varepsilon\rho\underline{t}(\varphi) + \Theta\varepsilon[\rho\underline{u}_{\mathrm{d}}(\varphi) + \underline{\hat{V}}_{\mathrm{p}}^{d}(y_1,y_2)] + \cdots \end{cases} \tag{55}$$

The index d is kept to recall the dependence on the choice of the particular solution $\underline{U}_{\mathrm{d}}$. On the one hand, a simplification occurs, $\underline{\hat{V}}^{2}(y_1, y_2) = 0$, the "T-stress" (52) and (53) is solution to the unperturbed as well as perturbed inner problems. On the other hand, the last term is completed by $\underline{\hat{V}}_{\mathrm{p}}^{d}(y_1, y_2)$ since $\underline{U}_{\mathrm{d}}$ does not fulfil the boundary conditions along the extension F_E.

Expression (46) of the change in stored energy still holds

$$\Delta W_{\mathrm{p}} = -1/2 \int_{F_E} \tilde{\sigma}(\underline{U}_{\mathrm{d}}^{\mathrm{h}0}) \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}\varepsilon}\, \mathrm{d}l - 1/2 \int_{F_E} [\tilde{\sigma}(\underline{U}_{\mathrm{d}}) - \beta I \Theta] \cdot \underline{n} \cdot \underline{U}^{\mathrm{h}\varepsilon} \mathrm{d}l,$$

with

$$\underline{U}^{\mathrm{h}\varepsilon}(x_1, x_2) = \underline{U}_{\mathrm{p}}^{\theta\varepsilon}(x_1, x_2) - \underline{U}_{\mathrm{d}}(x_1, x_2). \tag{56}$$

On the contrary, the index d is not kept here for simplicity reasons, in fact $\underline{U}^{\mathrm{h}\varepsilon}$ would have the indices p and d to recall that it is the penetration problem but with $\underline{U}_{\mathrm{d}}$ as particular solution. The first integral involves homogeneous terms (i.e. satisfying homogeneous boundary conditions in the vicinity of the primary crack tip) and can be transformed into a contour integral

$$\Delta W_{\mathrm{p}} = 1/2\int_C [\tilde{\sigma}(\underline{U}^{\mathrm{h}\varepsilon})\cdot\underline{n}\cdot\underline{U}^{\mathrm{h}0}_{\mathrm{d}} - \tilde{\sigma}(\underline{U}^{\mathrm{h}0}_{\mathrm{d}})\cdot\underline{n}\cdot\underline{U}^{\mathrm{h}\varepsilon}]\,\mathrm{d}l - T'_\theta\int_{F_E}\underline{U}^{\mathrm{h}\varepsilon}\cdot\underline{n}\,\mathrm{d}l$$
$$= \tilde{\psi}(\underline{U}^{\mathrm{h}\varepsilon},\underline{U}^{\mathrm{h}0}_{\mathrm{d}}) - T'_\theta\int_{F_E}\underline{U}^{\mathrm{h}\varepsilon}\cdot\underline{n}\,\mathrm{d}l, \tag{57}$$

where the constant T'_θ is defined by Eq. (54) and can be compared to Eq. (43)

$$T'_\theta = \frac{\sigma^{(2)}_{\mathrm{d}22}}{2} = \frac{1}{2(1-v)}\frac{E}{1+v}\Theta[\![\alpha]\!] = \tau'_\theta\frac{E}{1+v}\Theta[\![\alpha]\!], \tag{58}$$

It remains in Eq. (57) an integral along the extension F_E but it involves only a displacement field, namely the opening part of $\underline{U}^{\mathrm{h}\varepsilon}$, no stresses are required. The inner expansion (55) with (56) can be used to evaluate it

$$\underline{U}^{\mathrm{h}\varepsilon}(\varepsilon y_1,\varepsilon y_2) = \underline{U}^{\mathrm{h}0}_{\mathrm{p}}(0,0) + k_\theta\sqrt{\varepsilon}[\sqrt{\rho}\underline{u}(\varphi) + \underline{\hat{V}}^1_{\mathrm{p}}(y_1,y_2)] + T_{\theta\mathrm{d}}\varepsilon\rho\underline{t}(\varphi) + \Theta\varepsilon\underline{\hat{V}}^d_{\mathrm{p}}(y_1,y_2) + \cdots,$$

using continuity properties, it leads to

$$-\int_{F_E}\underline{U}^{\mathrm{h}\varepsilon}\underline{n}\,\mathrm{d}l = -k_\theta\varepsilon\sqrt{\varepsilon}\int_{F_E}\underline{\hat{V}}^1_{\mathrm{p}}\cdot\underline{n}\,\mathrm{d}L - \Theta\varepsilon^2\int_{F_E}\underline{\hat{V}}^d_{\mathrm{p}}\cdot\underline{n}\,\mathrm{d}L + \cdots = k_\theta\varepsilon\sqrt{\varepsilon}B + \Theta\varepsilon^2B' + \cdots,$$

where the integrals are calculated now on the stretched extension (still noted F_E) with $\mathrm{d}L = \mathrm{d}l/\varepsilon$. With these notations G (incremental) reads

$$G_{\mathrm{p}} = \frac{1}{\ell}(k_\theta^2K_{\mathrm{p}} + k_\theta T'_\theta B\sqrt{\varepsilon} + \cdots) \tag{59}$$

Indeed, $K'_{\mathrm{p}} = 0$ since $\underline{\hat{V}}^2 = 0$ and it can be proved that $H_{\mathrm{p}} = 0$ (Leguillon, 1993). Then the truncated criterion writes

$$\frac{G_{\mathrm{d}}}{G_{\mathrm{p}}} = \frac{K_{\mathrm{d}} + \eta^\theta_{\mathrm{d}}(K'_{\mathrm{d}} + H_{\mathrm{d}})}{K_{\mathrm{p}} + \eta^\theta_{\mathrm{p}}B} > \frac{G_{ic}}{G_{2c}},$$

with

$$\eta^\theta_{\mathrm{d}} = \frac{T_{\theta\mathrm{d}}}{k_\theta}\sqrt{\varepsilon_{\mathrm{d}}},\quad \eta^\theta_{\mathrm{p}} = \frac{T'_\theta}{k_\theta}\sqrt{\varepsilon_{\mathrm{p}}}.$$

The only term to calculate is (see Eq. (4))

$$B = -\int_{F_E}\underline{\hat{V}}^1_{\mathrm{p}}\cdot\underline{n}\,\mathrm{d}L = -\int_{F_E}\underline{V}^1_{\mathrm{p}}\cdot\underline{n}\,\mathrm{d}L,$$

which can be done analytically since $\underline{V}^1(y_1,y_2)$ has a simple form in case of homogeneous materials

$$\underline{V}^1_{\mathrm{p}}(y_1,y_2) = \sqrt{\rho'}\,\underline{u}(\varphi'),$$

where ρ' and φ' are new polar coordinates with origin at the tip of the crack extension (see Fig. 13). Then B is related to the mode I crack opening

$$B = [u_2(0) - u_2(2\pi)]\int_0^1\sqrt{\rho'}\,\mathrm{d}\rho' = \frac{2}{3}[u_2(0) - u_2(2\pi)].$$

Thus, considering Eqs. (43) and (58), the same previous conclusions can be drawn in this case. It must be simply pointed out that if the usual differential definition of the energy release rate is retained instead of Eq. (59), no asymptotics are finally required

$$G_{\mathrm{d}} = k_\theta^2K_{\mathrm{d}},\qquad G_{\mathrm{p}} = k_\theta^2K_{\mathrm{p}}.$$

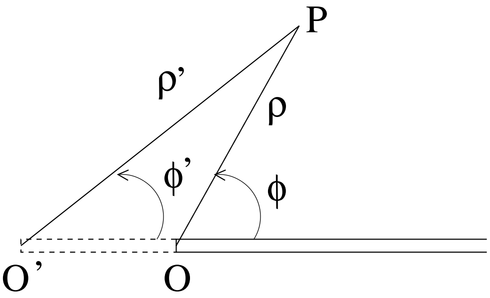

Fig. 13. The change of origin for polar coordinates after a crack extension.

The deflection criterion takes again the simplified form

$$\frac{G_{\mathrm{d}}}{G_{\mathrm{p}}} = \frac{K_{\mathrm{d}}}{K_{\mathrm{p}}} > \frac{G_{ic}}{G_{2c}},$$

and, of course, no deflection and penetration lengths are involved ($2\lambda - 1 = 0$).

6. Conclusion

From an asymptotic analysis, it is shown that, in composite materials, the main role of residual thermal stresses is to modify the stress intensity factors which split into a sum of a mechanical and a thermal contribution, whereas the criterion which governs the crack path selection remains unchanged from mechanical to thermal and to combined thermal and mechanical loadings. Thus, residual stresses influence the load level at which the mechanism starts but not the mechanism itself.

A secondary effect is to add a new term in the expansions, it is a non-singular one which occurs at the same order than the generalized "T-stress". Taking into account this additional term makes the criterion evolve from a local formulation (i.e. depending only on local properties around the primary crack tip) to a non-local one in which the geometry of the whole structure or specimen plays a role. If mechanical and thermal loadings are taken separately, the improved criterion remains independent of the intensity of the applied loads (i.e. mechanical forces or cooling temperature of a thermal process). This independence property disappears for a combined loading, the criterion makes reference to the actual mechanical load which triggers the crack growth or kinking, except in case of a very strong singularity when material 1 (containing the primary crack) is by far stiffer than material 2.

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